

EXPLICIT INVERSIONS OF CERTAIN MATRICES I

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ABSTRACT. In this note, we demonstrate a method to invert some Hankel matrices explicitly by using the kernel polynomials for the related classical orthogonal polynomials.

1. INTRODUCTION

In the theory of orthogonal polynomials, We could calculate the determinants of some Hankel matrices once we know the three term recurrence relation for the associated orthogonal polynomials and vice versa. It is well-known that the kernel polynomials of the orthogonal polynomials encodes important information about the associated Hankel matrices. These matrices are generalizations of the Hilbert matrices. In this note we present a method to invert some Hankel matrices associated with classical polynomials by using the kernel polynomials.

The following theorem very is a well known fact from the theory of orthogonal polynomials:

Theorem 1.1. *Given a probability measure μ on \mathbb{R} with a support of infinite many points. Let us consider the Hilbert space of μ -measurable functions*

$$(1.1) \quad \mathcal{X} := \left\{ f(x) \mid \int |f(x)|^2 d\mu(x) < \infty \right\}$$

with the inner product defined as

$$(1.2) \quad (f, g) := \int f(x) \overline{g(x)} d\mu(x), \quad f, g \in \mathcal{X}.$$

Assume that $\{w_n(x)\}_{n=0}^{\infty}$ is a sequence of linearly independent functions in \mathcal{X} with $w_0(x) = 1$. Let

$$(1.3) \quad \alpha_{jk} := \int w_j(x) \overline{w_k(x)} d\mu(x), \quad j, k = 0, 1, \dots,$$

$$(1.4) \quad \Pi_n := \begin{pmatrix} \alpha_{00} & \alpha_{01} & \dots & \alpha_{0n} \\ \alpha_{10} & \alpha_{11} & \dots & \alpha_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ \alpha_{n0} & \alpha_{n1} & \dots & \alpha_{nn} \end{pmatrix}, \quad n \in \mathbb{N} \cup \{0\},$$

and

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$$(1.5) \quad \Delta_n := \det \Pi_n, \quad n \in \mathbb{N} \cup \{0\}.$$

Then the n -th orthonormal function with positive coefficient in $w_n(x)$ is given by the formula

$$(1.6) \quad p_n(x) = \frac{1}{\sqrt{\Delta_n \Delta_{n-1}}} \det \begin{pmatrix} \alpha_{00} & \alpha_{01} & \alpha_{02} & \dots & \alpha_{0n} \\ \alpha_{10} & \alpha_{11} & \alpha_{12} & \dots & \alpha_{1n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \alpha_{n0} & \alpha_{n1} & \alpha_{n2} & \dots & \alpha_{nn} \\ w_0(x) & w_1(x) & w_2(x) & \dots & w_n(x) \end{pmatrix}$$

for $n \in \mathbb{N}$, with

$$(1.7) \quad p_0(x) = w_0(x) = 1.$$

Furthermore, the coefficient of $p_n(x)$ in $w_n(x)$ is

$$(1.8) \quad \gamma_n := \sqrt{\frac{\Delta_{n-1}}{\Delta_n}}.$$

Proof. The proof is the same as for the case $w_n(x) = x^n$, which could be found in any orthogonal polynomials textbooks such as [2]. \square

Corollary 1.2. For $n \in \mathbb{N}$, we have

$$(1.9) \quad \Delta_n = \prod_{k=1}^n \frac{1}{\gamma_k^2}.$$

Proof. This is a trivial consequence of (1.7) and (1.8). \square

Lemma 1.3. Let

$$(1.10) \quad k_n(x, y) := \sum_{k=0}^n p_k(x) \overline{p_k(y)}, \quad n \in \mathbb{N} \cup \{0\}.$$

Then, for any $\pi(x)$ in the linear span of $\{w_k(x)\}_0^n$, we have

$$(1.11) \quad \int \pi(x) \overline{k_n(x, y)} d\mu(x) = \pi(y).$$

Proof. To see (1.11), just expand $\pi(x)$ in $p_k(x)$, $k = 0, 1, \dots, n$. \square

Lemma 1.4. For each $n \in \mathbb{N} \cup \{0\}$, the function $k_n(x, y)$ satisfying (1.11) is unique.

Proof. Suppose there are two such functions $h_n(x, y)$ and $k_n(x, y)$, then,

$$(1.12) \quad \begin{aligned} 0 &< \|h_n(\cdot, y) - k_n(\cdot, y)\|^2 \\ &= (h_n(\cdot, y) - k_n(\cdot, y), h_n(\cdot, y) - k_n(\cdot, y)) \\ &= (h_n(\cdot, y) - k_n(\cdot, y), h_n(\cdot, y)) - (h_n(\cdot, y) - k_n(\cdot, y), k_n(\cdot, y)) \\ &= 0, \end{aligned}$$

which is a contradiction. \square

Lemma 1.5. *Let*

$$(1.13) \quad (\beta_{jk})_{0 \leq j, k \leq n} := \Pi_n^{-1}, \quad n \in \mathbb{N} \cup \{0\}.$$

Then,

$$(1.14) \quad k_n(x, y) = \sum_{j, k=0}^n \beta_{jk} \overline{w_j(y)} w_k(x).$$

Proof. Let

$$(1.15) \quad f(x) = \sum_{k=0}^n u_k w_k(x),$$

then,

$$(1.16) \quad \begin{aligned} & (f(\cdot), \sum_{j, k=0}^n \beta_{jk} \overline{w_j(y)} w_k(\cdot)) \\ &= \sum_{m=0}^n u_m (w_m(\cdot), k(\cdot, y)) \\ &= \sum_{m=0}^n u_m \sum_{j, k=0}^n \overline{\beta_{jk} w_j(y)} (w_m, w_k) \\ &= \sum_{m=0}^n u_m \sum_{j=0}^n w_j(y) \sum_{k=0}^n \overline{\beta_{jk} \alpha_{km}} \\ &= f(y). \end{aligned}$$

By Lemma 1.4, we have

$$(1.17) \quad k_n(x, y) = \sum_{j, k=0}^n \beta_{jk} \overline{w_j(y)} w_k(x).$$

□

Corollary 1.6. *The kernel in Lemma 1.3 is also given by*

$$(1.18) \quad k_n(x, y) = -\frac{1}{\Delta_n} \det \begin{pmatrix} 0 & 1 & \overline{w_1(y)} & \cdots & \overline{w_n(y)} \\ 1 & \alpha_{00} & \alpha_{01} & \cdots & \alpha_{0n} \\ w_1(x) & \alpha_{10} & \alpha_{11} & \cdots & \alpha_{1n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ w_n(x) & \alpha_{n0} & \alpha_{n1} & \cdots & \alpha_{nn} \end{pmatrix}$$

for $n \in \mathbb{N} \cup \{0\}$.

Proof. Since

$$(1.19) \quad k_n(x, y) = \sum_{j, k=0}^n \beta_{jk} \overline{w_j(y)} w_k(x),$$

with

$$(1.20) \quad (\beta_{jk})_{0 \leq i, j \leq n} = \Pi_n^{-1}.$$

Then,

$$(1.21) \quad \beta_{jk} = \frac{\Pi_n(k, j)}{\det \Pi_n} = \frac{\Pi_n(k, j)}{\Delta_n},$$

where $\Pi_n(k, j)$ is the (k, j) -th co-factor. Therefore,

$$(1.22) \quad k_n(x, y) = \frac{1}{\Delta_n} \sum_{j,k=0}^n \Pi_n(k, j) \overline{w_j(y)} w_k(x).$$

It is clear that

$$(1.23) \quad \sum_{j,k=0}^n \Pi_n(k, j) \overline{w_j(y)} w_k(x) = - \begin{vmatrix} 0 & (\overline{\mathbf{W}(\mathbf{y})})^T \\ \mathbf{W}(\mathbf{x}) & \Pi_n \end{vmatrix},$$

by direct determination expansion, which is

$$(1.24) \quad k_n(x, y) = -\frac{1}{\Delta_n} \begin{vmatrix} 0 & (\overline{\mathbf{W}(\mathbf{y})})^T \\ \mathbf{W}(\mathbf{x}) & \Pi_n \end{vmatrix},$$

where

$$(1.25) \quad \mathbf{W}(\mathbf{x}) = \begin{pmatrix} 1 \\ w_1(x) \\ \vdots \\ w_n(x) \end{pmatrix},$$

and

$$(1.26) \quad (\overline{\mathbf{W}(\mathbf{y})})^T = (1, \overline{w_1(y)}, \dots, \overline{w_n(y)}).$$

□

Lemma 1.5 enables us to compute the inverse the Gram matrix in terms of the orthonormal functions $\{p_n(x)\}_{n=0}^\infty$.

Corollary 1.7. *Assume that $\{w_n(x)\}_{n=0}^\infty$, $\{p_n(x)\}_{n=0}^\infty$ and $\Pi_n = (\alpha_{jk})_{0 \leq j, k \leq n}$ as in Theorem 1.1. Suppose we have two families of linear functionals $\{u_k\}_{k=0}^\infty$ and $\{v_k\}_{k=0}^\infty$ over the linear space generated by $\{w_n(x)\}_{n=0}^\infty$ with*

$$(1.27) \quad u_j(w_k) = \delta_{jk},$$

and

$$(1.28) \quad v_j(\overline{w_k}) = \delta_{jk}$$

for $j, k = 0, 1, \dots$. Then,

$$(1.29) \quad \beta_{jk} = \sum_{m=0}^n u_k(p_m(x)) v_j(\overline{p_m(y)}),$$

where

$$(1.30) \quad (\beta_{jk})_{0 \leq j, k \leq n} = \Pi_n^{-1}.$$

Proof. From Lemma 1.5, we have

$$(1.31) \quad \sum_{j,k=0}^n \beta_{jk} \overline{w_j(y)} w_k(x) = \sum_{m=0}^n \overline{p_m(y)} p_m(x).$$

Then we apply the functional u_j and v_k both sides of the above equation, the claim of the corollary follows. \square

2. MAIN RESULTS

2.1. Preliminaries. The Euler's $\Gamma(z)$ is defined as [1]

$$(2.1) \quad \frac{1}{\Gamma(z)} := z \prod_{j=1}^{\infty} \left(1 + \frac{z}{j}\right) \left(1 + \frac{1}{j}\right)^{-z}, \quad z \in \mathbb{C}$$

For $a, a_1, \dots, a_r \in \mathbb{C}$, the shifted factorials are defined as

$$(2.2) \quad (a)_n := \frac{\Gamma(a+n)}{\Gamma(a)}, \quad (a_1, \dots, a_r)_n := \prod_{j=1}^r (a_j)_n, \quad n \in \mathbb{Z}, r \in \mathbb{N}.$$

The generalized hypergeometric series ${}_rF_s$ with parameters $\{a_1, \dots, a_r\}$ and $\{b_1, \dots, b_s\}$ is formally defined by

$$(2.3) \quad {}_rF_s \left(\begin{matrix} a_1, a_2, \dots, a_r \\ b_1, b_2, \dots, b_s \end{matrix}; z \right) := \sum_{n=0}^{\infty} \frac{(a_1, \dots, a_r)_n}{(a_1, \dots, a_s)_n} \frac{z^n}{n!}.$$

The Barnes G -function is defined as

$$(2.4) \quad G(z) := (2\pi)^{z/2} e^{-[z(z+1)+\gamma z^2]/2} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right)^n e^{-z+z^2/(2n)},$$

where

$$(2.5) \quad \gamma := \lim_{n \rightarrow \infty} \left(\sum_{k=1}^n \frac{1}{k} - \ln n \right).$$

The Barnes G -function is an entire function with the property

$$(2.6) \quad G(z+1) = \Gamma(z)G(z),$$

$$(2.7) \quad \prod_{k=0}^n \Gamma(z+k) = \frac{G(z+n+1)}{G(z)},$$

and

$$(2.8) \quad G(n) = \begin{cases} 0 & n = 0, -1, -2, \dots \\ \prod_{i=0}^{n-2} i! & n = 1, 2, \dots \end{cases}.$$

In following, in all the cases except the last one we use functionals

$$(2.9) \quad u_i(p(x)) = v_i(p(x)) = \frac{1}{i!} \left[\frac{d^i p(x)}{dx^i} \right]_{x=0},$$

and for the last case we use

$$(2.10) \quad u_i(p(x)) = v_i(p(x)) = \frac{1}{i!} \left[\frac{d^i p(x)}{dx^i} \right]_{x=1},$$

where $p(x)$ is a polynomial in variable x .

2.2. The Hermite Polynomials. The Hermite polynomials $\{H_n(x)\}_{n=0}^\infty$ are defined as [1]

$$(2.11) \quad H_n(x) = (2x)^n {}_2F_0 \left(\begin{matrix} -\frac{n}{2}, -\frac{n}{2} + \frac{1}{2} \\ - \end{matrix} ; -\frac{1}{x^2} \right)$$

for $n \geq 0$ and

$$(2.12) \quad H_{-1}(x) = 0.$$

They satisfy

$$(2.13) \quad DH_n(x) = 2nH_{n-1}(x), \quad n \in \mathbb{N} \cup \{0\}.$$

Hermite polynomials satisfies

$$(2.14) \quad \int_{\mathbb{R}} H_n(x) H_m(x) \exp(-x^2) dx = 2^n n! \sqrt{\pi} \delta_{mn}$$

for $n, m = 0, 1, \dots$

Thus, the orthonormal polynomials

$$(2.15) \quad h_n(x) := \frac{H_n(x)}{\sqrt{n! 2^n \sqrt{\pi}}}$$

have leading coefficients

$$(2.16) \quad \gamma_n = \sqrt{\frac{2^n}{n! \sqrt{\pi}}}.$$

Clearly,

$$(2.17) \quad \int_{-\infty}^{\infty} y^n e^{-y^2} dy = \frac{1 + (-1)^n}{2} \Gamma\left(\frac{n+1}{2}\right),$$

and

$$(2.18) \quad \alpha_{ij} = \frac{1 + (-1)^{i+j}}{2} \Gamma\left(\frac{i+j+1}{2}\right),$$

for $i, j = 0, 1, \dots, n$. Thus,

$$(2.19) \quad \det \left(\frac{1 + (-1)^{i+j}}{2} \Gamma\left(\frac{i+j+1}{2}\right) \right)_{j,k=0}^n = 2^{-\frac{n(n+1)}{2}} \pi^{\frac{n+1}{2}} \prod_{k=0}^n k!$$

or

$$(2.20) \quad \det \left(\frac{1 + (-1)^{i+j}}{2} \Gamma\left(\frac{i+j+1}{2}\right) \right)_{j,k=0}^n = 2^{-\frac{n(n+1)}{2}} \pi^{\frac{n+1}{2}} G(n+2)$$

for $n = 0, 1, \dots$

The (i, j) -th entry of $\Pi_n^{-1} = (\beta_{jk})_{j,k=0}^n$ is

$$(2.21) \quad \begin{aligned} \beta_{ij} &= \frac{1}{i!j!} \sum_{k=\max(i,j)}^n \frac{1}{k! 2^k \sqrt{\pi}} \left[\frac{d^i H_k(x)}{dx^i} \right]_{x=0} \left[\frac{d^j H_k(y)}{dy^j} \right]_{y=0} \\ &= \sum_{k=\max(i,j)}^n \frac{1}{k! 2^k \sqrt{\pi}} \left\{ 2^i \binom{k}{i} H_{k-i}(0) \right\} \left\{ 2^j \binom{k}{j} H_{k-j}(0) \right\} \end{aligned}$$

or

$$(2.22) \quad \beta_{ij} = \frac{2^{i+j}}{\sqrt{\pi}} \sum_{k=\max(i,j)}^n \frac{1}{k!2^k} \left\{ \binom{k}{i} H_{k-i}(0) \right\} \left\{ \binom{k}{j} H_{k-j}(0) \right\}.$$

for $i, j = 0, 1, \dots, n$.

Theorem 2.1. *For $n \in \mathbb{N} \cup \{0\}$, the matrix*

$$(2.23) \quad \left(\frac{1 + (-1)^{i+j}}{2\sqrt{\pi}} \Gamma\left(\frac{i+j+1}{2}\right) \right)_{0 \leq i, j \leq n}$$

has the determinant

$$(2.24) \quad \det \left(\frac{1 + (-1)^{i+j}}{2\sqrt{\pi}} \Gamma\left(\frac{i+j+1}{2}\right) \right)_{i,j=0}^n = 2^{-\frac{n(n+1)}{2}} G(n+2),$$

and its inverse matrix is

$$(2.25) \quad \left(\sum_{k=\max(i,j)}^n \frac{2^{i+j} \left\{ \binom{k}{i} H_{k-i}(0) \right\} \left\{ \binom{k}{j} H_{k-j}(0) \right\}}{k!2^k} \right)_{0 \leq i, j \leq n}.$$

2.2.1. *The Laguerre Polynomials.* The Laguerre polynomials $\{L_n^\alpha(x)\}_{n=0}^\infty$ may be defined as [1]

$$(2.26) \quad L_n^\alpha(x) = \frac{(\alpha+1)_n}{n!} {}_1F_1 \left(\begin{matrix} -n \\ \alpha+1 \end{matrix}; x \right)$$

for $n \geq 0$, and we assume that

$$(2.27) \quad L_{-1}^\alpha(x) = 0.$$

We also have

$$(2.28) \quad \frac{dL_n^\alpha(x)}{dx} = -L_{n-1}^{\alpha+1}(x)$$

and

$$(2.29) \quad \int_0^\infty L_m^\alpha(x) L_n^\alpha(x) x^\alpha e^{-x} dx = \frac{\Gamma(\alpha+n+1)}{n!} \delta_{mn},$$

for $\alpha > -1$ and $n, m = 0, 1, \dots$. The orthonormal polynomials

$$(2.30) \quad p_n(x) = (-1)^n \sqrt{\frac{n!}{\Gamma(\alpha+n+1)}} L_n^\alpha(x),$$

have the leading coefficients

$$(2.31) \quad \gamma_n = \frac{1}{\sqrt{n! \Gamma(\alpha+n+1)}}$$

for $n = 0, 1, \dots$. Clearly

$$(2.32) \quad \int_0^\infty x^{n+\alpha} e^{-x} dx = \Gamma(\alpha+n+1),$$

and

$$(2.33) \quad \alpha_{ij} = \Gamma(\alpha + i + j + 1)$$

for $i, j = 0, 1, \dots, n$, Then,

$$(2.34) \quad \det (\Gamma(\alpha + i + j + 1))_{j,k=0}^n = \prod_{k=0}^n \{k! \Gamma(\alpha + k + 1)\},$$

or

$$(2.35) \quad \det (\Gamma(\alpha + i + j + 1))_{j,k=0}^n = \frac{G(n+2)G(\alpha+n+2)}{G(\alpha+1)}.$$

Let $\Pi_n^{-1} = (\beta_{jk})_{j,k=0}^n$, then,

$$(2.36) \quad \begin{aligned} \beta_{ij} &= \frac{1}{i!j!} \sum_{k=\max(i,j)}^n \frac{k!}{\Gamma(\alpha+k+1)} \left[\frac{d^i L_k^\alpha(x)}{dx^i} \right]_{x=0} \left[\frac{d^j L_k^\alpha(y)}{dy^j} \right]_{y=0} \\ &= \frac{1}{i!j!} \sum_{k=\max(i,j)}^n \frac{k!}{\Gamma(\alpha+k+1)} [(-1)^i L_{k-i}^{\alpha+i}(x)]_{x=0} [(-1)^j L_{k-j}^{\alpha+j}(y)]_{y=0}, \end{aligned}$$

or

$$(2.37) \quad \beta_{ij} = \frac{(-1)^{i+j}}{i!j!} \sum_{k=\max(i,j)}^n \frac{k! L_{k-i}^{\alpha+i}(0) L_{k-j}^{\alpha+j}(0)}{\Gamma(\alpha+k+1)}$$

for $j, k = 0, 1, \dots, n$.

Theorem 2.2. For $n = 0, 1, \dots$, the matrix

$$(2.38) \quad ((\alpha+1)_{i+j})_{0 \leq i,j \leq n}$$

has determinant

$$(2.39) \quad \det ((\alpha+1)_{i+j})_{i,j=0}^n = \frac{G(n+2)G(\alpha+n+2)}{G(\alpha+1)\Gamma(\alpha+1)^{n+1}},$$

and inverse

$$(2.40) \quad \left(\frac{\sum_{k=\max(i,j)}^n \frac{(\alpha+1)_k}{k!} \binom{k}{i} \binom{k}{j}}{(-1)^{i+j} (\alpha+1)_i (\alpha+1)_j} \right)_{0 \leq i,j \leq n}$$

2.2.2. The Ultraspherical Polynomials. The Ultraspherical polynomials (or Gegenbauer polynomials) $\{C_n^\lambda(x)\}_{n=0}^\infty$ are defined as, [1]

$$(2.41) \quad C_n^\lambda(x) = \frac{(2\lambda)_n}{n!} {}_2F_1 \left(\begin{matrix} -n, 2\lambda+n \\ \lambda+\frac{1}{2} \end{matrix}; \frac{1-x}{2} \right)$$

for $n \geq 0$, and we assume that

$$(2.42) \quad C_{-1}^\lambda(x) = 0.$$

We also have

$$(2.43) \quad \frac{dC_n^\lambda(x)}{dx} = 2\lambda C_{n-1}^{\lambda+1}(x),$$

$$(2.44) \quad \int_{-1}^1 C_m^\lambda(x) C_n^\lambda(x) (1-x^2)^{\lambda-\frac{1}{2}} dx = \frac{\pi \Gamma(2\lambda+n)}{2^{2\lambda-1} n! (\lambda+n) [\Gamma(\lambda)]^2} \delta_{mn},$$

for $\lambda > -\frac{1}{2}$ and $n, m = 0, 1, \dots$. The orthonormal polynomials

$$(2.45) \quad p_n(x) = \sqrt{\frac{2^{2\lambda-1}n!(\lambda+n)[\Gamma(\lambda)]^2}{\pi\Gamma(2\lambda+n)}} C_n^\lambda(x)$$

have leading coefficients

$$(2.46) \quad \gamma_n = \sqrt{\frac{(\lambda+n)2^{2\lambda+2n-1}\Gamma(\lambda+n)^2}{\pi n!\Gamma(2\lambda+n)}}.$$

It is clear that

$$(2.47) \quad \int_{-1}^1 x^n (1-x^2)^{\lambda-\frac{1}{2}} dx = \frac{1+(-1)^n}{2} B\left(\frac{n+1}{2}, \lambda + \frac{1}{2}\right),$$

and

$$(2.48) \quad \alpha_{ij} = \frac{1+(-1)^{i+j}}{2} B\left(\frac{i+j+1}{2}, \lambda + \frac{1}{2}\right),$$

for $i, j = 0, 1, \dots, n$, where $B(p, q)$ is the beta integral

$$(2.49) \quad B(p, q) = \int_0^1 x^{p-1} (1-x)^{q-1} dx, \quad \Re(p), \Re(q) > 0.$$

Then,

$$(2.50) \quad \det \left(\frac{1+(-1)^{i+j}}{2} B\left(\frac{i+j+1}{2}, \lambda + \frac{1}{2}\right) \right) = \prod_{k=0}^n \frac{\pi k! \Gamma(2\lambda+k)}{(\lambda+k) 2^{2\lambda+2k-1} \Gamma(\lambda+k)^2},$$

or

$$(2.51) \quad \det \left(\frac{1+(-1)^{i+j}}{2} B\left(\frac{i+j+1}{2}, \lambda + \frac{1}{2}\right) \right) = \frac{\pi^{n+1} G(n+2)}{2^{(n+1)(n+2\lambda-1)} (\lambda)_{n+1}} \frac{G(2\lambda+n+1) G(\lambda)^2}{G(2\lambda) G(\lambda+n+1)^2}.$$

The (i, j) -th entry of the inverse matrix $\Pi_n^{-1} = (\beta_{jk})_{j,k=0}^n$ is

$$(2.52) \quad \begin{aligned} \beta_{ij} &= \frac{1}{i!j!} \sum_{k=\max(i,j)}^n \frac{2^{2\lambda-1} k! (\lambda+k) [\Gamma(\lambda)]^2}{\pi \Gamma(2\lambda+k)} \left[\frac{d^i C_k^\lambda(x)}{dx^i} \right]_{x=0} \left[\frac{d^j C_k^\lambda(y)}{dy^j} \right]_{y=0} \\ &= \frac{2^{2\lambda-1} [\Gamma(\lambda)]^2}{i!j!\pi} \sum_{k=\max(i,j)}^n \frac{k! (\lambda+k)}{\Gamma(2\lambda+k)} \left[2^i (\lambda)_i C_{k-i}^{\lambda+i}(x) \right]_{x=0} \left[2^j (\lambda)_j C_{k-j}^{\lambda+j}(x) \right]_{y=0}, \end{aligned}$$

or

$$(2.53) \quad \beta_{ij} = \frac{2^{i+j} (\lambda)_i (\lambda)_j \Gamma(\lambda)}{i!j! \sqrt{\pi} \Gamma(\lambda + \frac{1}{2})} \sum_{k=\max(i,j)}^n \frac{k! (\lambda+k) C_{k-i}^{\lambda+i}(0) C_{k-j}^{\lambda+j}(0)}{(2\lambda)_k}$$

for $i, j = 0, 1, \dots, n$.

Theorem 2.3. *Let $(\alpha_{ij})_{0 \leq i, j \leq n}$ be the matrix with entries*

$$(2.54) \quad \alpha_{ij} = \frac{1 + (-1)^{i+j}}{2} B\left(\frac{i+j+1}{2}, \lambda + \frac{1}{2}\right)$$

for $i, j = 0, 1, \dots, n$, then,

$$(2.55) \quad \det\left(\frac{1 + (-1)^{i+j}}{2} B\left(\frac{i+j+1}{2}, \lambda + \frac{1}{2}\right)\right) = \prod_{k=0}^n \frac{\pi k! \Gamma(2\lambda + k)}{(\lambda + k) 2^{2\lambda+2k-1} \Gamma(\lambda + k)^2},$$

or

$$(2.56) \quad \det\left(\frac{1 + (-1)^{i+j}}{2} B\left(\frac{i+j+1}{2}, \lambda + \frac{1}{2}\right)\right) = \frac{\pi^{n+1} G(n+2)}{2^{(n+1)(n+2\lambda-1)} (\lambda)_{n+1}} \frac{G(2\lambda + n+1) G(\lambda)^2}{G(2\lambda) G(\lambda + n+1)^2}.$$

The inverse matrix $(\beta_{ij})_{0 \leq i, j \leq n}$ has entries

$$(2.57) \quad \beta_{ij} = \frac{2^{i+j}(\lambda)_i(\lambda)_j \Gamma(\lambda)}{i!j! \sqrt{\pi} \Gamma(\lambda + \frac{1}{2})} \sum_{k=\max(i,j)}^n \frac{k! (\lambda + k) C_{k-i}^{\lambda+i}(0) C_{k-j}^{\lambda+j}(0)}{(2\lambda)_k}$$

for $i, j = 0, 1, \dots, n$.

2.2.3. *The Jacobi Polynomials.* The Jacobi polynomials $\left\{P_n^{(\alpha, \beta)}(x)\right\}_{n=0}^{\infty}$ may be defined as [1, 2]

$$(2.58) \quad P_n^{(\alpha, \beta)}(x) = \frac{(\alpha + 1)_n}{n!} {}_2F_1\left(\begin{matrix} -n; n + \alpha + \beta + 1 \\ \alpha + 1 \end{matrix}; \frac{1-x}{2}\right)$$

for $n \geq 0$, and

$$(2.59) \quad P_{-1}^{(\alpha, \beta)}(x) = 0.$$

We also have

$$(2.60) \quad \frac{dP_n^{(\alpha, \beta)}(x)}{dx} = \frac{n + \alpha + \beta + 1}{2} P_{n-1}^{(\alpha+1, \beta+1)}(x),$$

and

$$(2.61) \quad \int_{-1}^1 P_m^{(\alpha, \beta)}(x) P_n^{(\alpha, \beta)}(x) w(x) dx = h_n \delta_{mn}$$

for $\alpha, \beta > -1$ and $n, m = 0, 1, \dots$ with

$$(2.62) \quad w(x) := (1-x)^\alpha (1+x)^\beta,$$

and

$$(2.63) \quad h_n := \frac{2^{\alpha+\beta+1} \Gamma(\alpha + n + 1) \Gamma(\beta + n + 1)}{(2n + \alpha + \beta + 1) \Gamma(\alpha + \beta + n + 1) n!}.$$

The orthonormal polynomials

$$(2.64) \quad p_n(x) = \sqrt{\frac{(2n + \alpha + \beta + 1) \Gamma(\alpha + \beta + n + 1) n!}{2^{\alpha+\beta+1} \Gamma(\alpha + n + 1) \Gamma(\beta + n + 1)}} P_n^{(\alpha, \beta)}(x)$$

with leading coefficients

$$(2.65) \quad \gamma_n = \frac{2^{n+(\alpha+\beta)/2} \Gamma\left(\frac{\alpha+\beta+1}{2} + n\right) \Gamma\left(\frac{\alpha+\beta+2}{2} + n\right) \sqrt{\frac{\alpha+\beta+1}{2} + n}}{\sqrt{n! \pi \Gamma(\alpha + \beta + n + 1) \Gamma(\alpha + n + 1) \Gamma(\beta + n + 1)}}$$

for $n = 0, 1, \dots$. The moments of the Jacobi measure are

$$(2.66) \quad \mu_n = \frac{2^{\alpha+\beta+1} \Gamma(\alpha + 1) \Gamma(\beta + 1)}{(-1)^n \Gamma(\alpha + \beta + 1)} {}_2F_1\left(\begin{matrix} -n, \beta + 1 \\ \alpha + \beta + 1 \end{matrix}; 2\right)$$

for $n = 0, 1, \dots$. Then (i, j) -th entry of the Hankel matrix $\Pi_n = (\alpha_{jk})_{j,k=0}^n$ is

$$(2.67) \quad \alpha_{ij} = \frac{(-1)^{i+j} \Gamma(\alpha + 1) \Gamma(\beta + 1)}{2^{-\alpha-\beta-1} \Gamma(\alpha + \beta + 1)} {}_2F_1\left(\begin{matrix} -i-j, \beta + 1 \\ \alpha + \beta + 1 \end{matrix}; 2\right)$$

for $i, j = 0, 1, \dots, n$. Then,

$$(2.68) \quad \det \Pi_n = \frac{\left(\frac{\pi}{2^{n+\alpha+\beta}}\right)^{n+1} G(n+2) G\left(\frac{\alpha+\beta+1}{2}\right)^2 G\left(\frac{\alpha+\beta+2}{2}\right)^2}{\left(\frac{\alpha+\beta+1}{2}\right)_{n+1} G\left(\frac{\alpha+\beta+3}{2} + n\right)^2 G\left(\frac{\alpha+\beta+4}{2} + n\right)^2} \\ \times \frac{G(\alpha + \beta + n + 2) G(\alpha + n + 2) G(\beta + n + 2)}{G(\alpha + \beta + 1) G(\alpha + 1) G(\beta + 1)},$$

and the (i, j) -th entry of the inverse matrix Π_n^{-1} is

$$(2.69) \quad \beta_{ij} = \frac{1}{i!j!} \sum_{k=\max(i,j)}^n \frac{(2k + \alpha + \beta + 1) \Gamma(\alpha + \beta + k + 1) k!}{2^{\alpha+\beta+1} \Gamma(\alpha + k + 1) \Gamma(\beta + k + 1)} \\ \times \left[\frac{d^i P_k^{(\alpha, \beta)}(x)}{dx^i} \right]_{x=0} \left[\frac{d^j P_k^{(\alpha, \beta)}(y)}{dy^j} \right]_{y=0} \\ = \frac{1}{i!j!} \sum_{k=\max(i,j)}^n \frac{(2k + \alpha + \beta + 1) \Gamma(\alpha + \beta + k + 1) k!}{2^{\alpha+\beta+1} \Gamma(\alpha + k + 1) \Gamma(\beta + k + 1)} \\ \times \left[\frac{(k + \alpha + \beta + 1)_i P_{k-i}^{(\alpha+i, \beta+i)}(x)}{2^i} \right]_{x=0} \left[\frac{(k + \alpha + \beta + 1)_j P_{k-j}^{(\alpha+j, \beta+j)}(y)}{2^j} \right]_{y=0},$$

or

$$(2.70) \quad \beta_{ij} = \sum_{k=\max(i,j)}^n \frac{(2k + \alpha + \beta + 1) \Gamma(\alpha + \beta + k + 1) k!}{\Gamma(\alpha + k + 1) \Gamma(\beta + k + 1)} \\ \times \left\{ \frac{(k + \alpha + \beta + 1)_i P_{k-i}^{(\alpha+i, \beta+i)}(0) \{(k + \alpha + \beta + 1)_j P_{k-j}^{(\alpha+j, \beta+j)}(0)\}}{i!j! 2^{\alpha+\beta+i+j+1}} \right\}.$$

for $i, j = 0, 1, \dots, n$.

Theorem 2.4. For $n = 0, 1, \dots$, the matrix

$$(2.71) \quad \left({}_2F_1\left(\begin{matrix} -i-j, \beta + 1 \\ \alpha + \beta + 1 \end{matrix}; 2\right) \right)_{0 \leq i, j \leq n}$$

has the determinant

$$\begin{aligned}
 (2.72) \quad & \det \left({}_2F_1 \left(\begin{matrix} -i-j, \beta+1 \\ \alpha+\beta+1 \end{matrix}; 2 \right) \right)_{j,k=0}^n \\
 &= \left(\frac{\Gamma(\alpha+\beta+1)2^{-(2\alpha+2\beta+n+1)\pi}}{\Gamma(\alpha+\beta+1)\Gamma(\alpha+\beta+1)} \right)^{n+1} \\
 &\times \frac{G(n+2)G\left(\frac{\alpha+\beta+1}{2}\right)^2 G\left(\frac{\alpha+\beta+2}{2}\right)^2}{G\left(\frac{\alpha+\beta+3}{2}+n\right)^2 G\left(\frac{\alpha+\beta+4}{2}+n\right)^2} \\
 &\times \frac{G(\alpha+\beta+n+2)G(\alpha+n+2)G(\beta+n+2)}{\left(\frac{\alpha+\beta+1}{2}\right)_{n+1} G(\alpha+\beta+1)G(\alpha+1)G(\beta+1)},
 \end{aligned}$$

and its inverse matrix $(\beta_{ij})_{0 \leq i,j \leq n}$ with

$$\begin{aligned}
 (2.73) \quad \beta_{ij} &= \sum_{k=\max(i,j)}^n \frac{k!(2k+\alpha+\beta+1)(\alpha+\beta+1)_k}{(-2)^{i+j}i!j!(\alpha+1)_k(\beta+1)_k} \\
 &\times \left\{ (k+\alpha+\beta+1)_i P_{k-i}^{(\alpha+i,\beta+i)}(0) \right\} \left\{ (k+\alpha+\beta+1)_j P_{k-j}^{(\alpha+j,\beta+j)}(0) \right\}.
 \end{aligned}$$

for $i, j = 0, 1, \dots, n$.

If we take the polynomial sequence

$$(2.74) \quad w_n(x) = (x-1)^n$$

for $n = 0, 1, \dots$, and the linear functionals defined in (2.10). Then the (i, j) -th entry of $\Pi_n = (\alpha_{jk})_{j,k=0}^n$ is

$$(2.75) \quad \alpha_{ij} = \int_{-1}^1 (x-1)^{i+j} w(x) dx$$

or

$$(2.76) \quad \alpha_{ij} = \frac{2^{\alpha+\beta+i+j+1}\Gamma(\alpha+i+j+1)\Gamma(\beta+1)}{(-1)^{i+j}\Gamma(\alpha+\beta+i+j+2)}$$

for $i, j = 0, 1, \dots, n$, and its determinant is given by

$$\begin{aligned}
 (2.77) \quad \det \Pi_n &= \frac{G(\alpha+n+1)G(\beta+n+1)G(\alpha+\beta+n+1)}{2^{(n+\alpha+\beta)(n+1)}G(\alpha+1)G(\beta+1)G(\alpha+\beta+1)} \\
 &\times \frac{\pi^{n+1}G(n+2)G\left(\frac{\alpha+\beta+1}{2}\right)^2 G\left(\frac{\alpha+\beta+2}{2}\right)^2}{\left(\frac{\alpha+\beta+1}{2}\right)_{n+1} G\left(\frac{\alpha+\beta+1}{2}+n+1\right)^2 G\left(\frac{\alpha+\beta+2}{2}+n+1\right)^2},
 \end{aligned}$$

and the it inverse matrix has entries

(2.78)

$$\begin{aligned}
\beta_{ij} &= \frac{1}{i!j!} \sum_{k=\max(i,j)}^n \frac{(2k+\alpha+\beta+1)\Gamma(\alpha+\beta+k+1)k!}{2^{\alpha+\beta+1}\Gamma(\alpha+k+1)\Gamma(\beta+k+1)} \\
&\quad \times \left[\frac{d^i P_k^{(\alpha,\beta)}(x)}{dx^i} \right]_{x=1} \left[\frac{d^j P_k^{(\alpha,\beta)}(y)}{dy^j} \right]_{y=1} \\
&= \sum_{k=\max(i,j)}^n \frac{(2k+\alpha+\beta+1)\Gamma(\alpha+\beta+k+1)k!}{2^{\alpha+\beta+1}\Gamma(\alpha+k+1)\Gamma(\beta+k+1)} \\
&\quad \times \left[\frac{(k+\alpha+\beta+1)_i P_{k-i}^{(\alpha+i,\beta+i)}(x)}{i!2^i} \right]_{x=1} \left[\frac{(k+\alpha+\beta+1)_j P_{k-j}^{(\alpha+j,\beta+j)}(y)}{j!2^j} \right]_{y=1},
\end{aligned}$$

or

$$\begin{aligned}
(2.79) \quad \beta_{ij} &= \sum_{k=\max(i,j)}^n \frac{(2k+\alpha+\beta+1)\Gamma(\alpha+\beta+k+1)k!}{\Gamma(\alpha+k+1)\Gamma(\beta+k+1)} \\
&\quad \times \left\{ \frac{(k+\alpha+\beta+1)_i P_{k-i}^{(\alpha+i,\beta+i)}(1) \{(k+\alpha+\beta+1)_j P_{k-j}^{(\alpha+j,\beta+j)}(1)\}}{i!j!2^{\alpha+\beta+i+j+1}} \right\}
\end{aligned}$$

for $i, j = 0, 1, \dots, n$. Since

$$(2.80) \quad P_n^{(\alpha,\beta)}(1) = \frac{(\alpha+1)_n}{n!},$$

then

$$\begin{aligned}
(2.81) \quad \beta_{ij} &= \frac{\Gamma(\alpha+\beta+1)(\alpha+\beta+1)_i(\alpha+\beta+1)_j}{2^{\alpha+\beta+i+j+1}(\alpha+1)_i(\alpha+1)_j\Gamma(\alpha+1)\Gamma(\beta+1)} \\
&\quad \times \sum_{k=\max(i,j)}^n \left\{ \frac{(2k+\alpha+\beta+1)(\alpha+1)_k}{k!(\alpha+\beta+1)_k(\beta+1)_k} \right\} \\
&\quad \times \left\{ \binom{k}{i} \binom{k}{j} (\alpha+\beta+i+1)_k (\alpha+\beta+j+1)_k \right\}
\end{aligned}$$

for $i, j = 0, 1, \dots, n$. Therefore, we have proved the following:

Theorem 2.5. *For $n = 0, 1, \dots$, the determinant of the matrix*

$$(2.82) \quad \left(\frac{(\alpha+1)_{i+j}}{(\alpha+\beta+2)_{i+j}} \right)_{0 \leq i, j \leq n}$$

is

$$\begin{aligned}
 (2.83) \quad & \det \left(\frac{(\alpha+1)_{i+j}}{(\alpha+\beta+2)_{i+j}} \right)_{i,j=0}^n \\
 &= \frac{G\left(\frac{\alpha+\beta+1}{2}\right)^2 G\left(\frac{\alpha+\beta+2}{2}\right)^2}{G(\alpha+1)G(\beta+1)G(\alpha+\beta+1)} \\
 & \times \left(\frac{\pi\Gamma(\alpha+\beta+2)}{2^{2n+2\alpha+2\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1)} \right)^{n+1} \\
 & \times \frac{G(n+2)G(\alpha+n+1)G(\beta+n+1)G(\alpha+\beta+n+1)}{\left(\frac{\alpha+\beta+1}{2}\right)_{n+1} G\left(\frac{\alpha+\beta+1}{2}+n+1\right)^2 G\left(\frac{\alpha+\beta+2}{2}+n+1\right)^2},
 \end{aligned}$$

and its inverse matrix $(\gamma_{ij})_{0 \leq i,j \leq n}$ has elements

$$\begin{aligned}
 (2.84) \quad & \gamma_{ij} = \frac{(-1)^{i+j}(\alpha+\beta+1)_i(\alpha+\beta+1)_j}{(\alpha+1)_i(\alpha+1)_j(\alpha+\beta+1)} \\
 & \times \sum_{k=\max(i,j)}^n \left\{ \frac{(2k+\alpha+\beta+1)(\alpha+1)_k}{k!(\alpha+\beta+1)_k(\beta+1)_k} \right\} \\
 & \times \left\{ \binom{k}{i} \binom{k}{j} (\alpha+\beta+i+1)_k (\alpha+\beta+j+1)_k \right\}
 \end{aligned}$$

for $i, j = 0, 1, \dots, n$.

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